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Market Managed Multiservice Internet

Competition in the Internet and dynamic pricing by ECN marks

Bjørn Hansen
and
Eric Nævdal
Telenor R&D
bjorn.hansen@telenor.com

Outline

- A modification of the ECN pricing scheme proposed by Kelly et. al.
- Stability
- Inelastic traffic
- The monopoly solution
- Perfect competition
- Externalities

Kelly:
Mathematical modelling of the Internet

$$\frac{d}{dy}C_j(y) = p_j(y)$$

•The pricing rule in resource ***j*** at load ***y***

$$\frac{d}{dt}x_r(t) = \kappa_r\left(w_r(t) - x_r(t)\sum_{j\in r}\mu_j(t)\right)$$

•The rate control algorithm

$$\mu_j(t) = p_j\left(\sum_{s:j\in s}x_s(t)\right)$$

•Unit price of traversing resource ***j***

$$w_r(t) = x_r(t)U_r'(x_r(t))$$

•Optimal "weight"

Kelly's theorem 2.2:

$$W(x) = \sum_{r\in R}U_r(x_r) - \sum_{j\in J}C_j\left(\sum_{s:j\in s}x_s\right)$$

is a Lyapunov function for the system, the unique value maximising $W(x)$ is accordingly a stable point of the system

Reformulation (1) utility maximisation

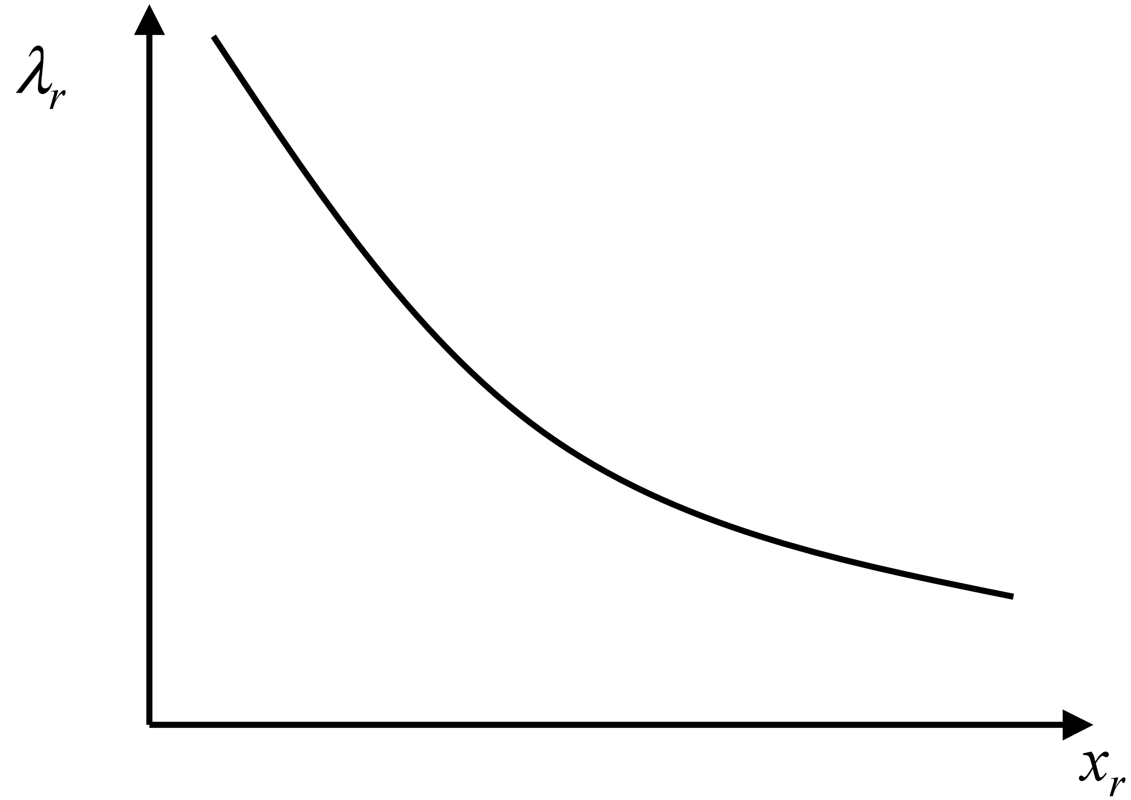
- Consider: $w_r(t) = x_r(t)U_r'(x_r(t))$
- It's the solution of: $\max[U(x) - w]$
- i.e. maximising utility of flow $\boldsymbol{x}$ minus cost $\boldsymbol{w}$
- Equivalent to: $\max(U_r(x_r) - \lambda_r x_r)$ where $\lambda_r \equiv \frac{w_r}{x_r}$
- Thus the users optimisation problem can be written:

$$\max_{x_r}(U_r(x_r) - \lambda_r x_r)$$

- i.e. maximise utility minus cost in the same way as above.

Reformulation (2) the demand function

$$x^*_r(\lambda_r) = \arg\max_{x_r}\left(U_r(x_r) - \lambda_r x_r\right), \quad \text{FoC:} \qquad \frac{\partial U_r(x_r)}{\partial x_r} = \lambda$$



Reformulation (3)
rate control

λ is the unit price the user is facing when traversing route r. This unit price is the number of marks received along the route:

$$\lambda_r = \sum_{j \in r} \mu_j(t) = \sum_{j \in r} \left[p_j \left(\sum_{s: j \in s} x_s(t) \right) \right]$$

Then we can replace w in the rate control differential equation since: $w_r = \lambda_r x^*_r$

$$\frac{d}{dt} x_r(t) = \kappa_r \left(w_r - x_r \sum_{j \in r} \mu_j \right) = \kappa_r \left(\lambda_r x_r^* - x_r \lambda_r \right) = \kappa_r \lambda_r \left(x_r^*(t) - x_r(t) \right)$$

Reformulation (4) the system

Kelly's system is equivalent to:

$$(2.1)\quad \frac{d}{dy}C_j(y) = p_j(y)$$

$$(2.2')\quad \frac{d}{dt}x_r(t) = \kappa_r \lambda_r(t)\left[x_r^{\ *}(t) - x_r(t)\right]$$

$$(2.3)\quad \lambda_r(t) = \sum_{j \in r}\left[p_j\left(\sum_{s: j \in s} x_s(t)\right)\right]$$

$$(2.6')\quad x^*{}_r(\lambda_r(t)) = \arg\max_{x_r}\left(U_r(x_r) - \lambda_r(t)x_r\right)$$

Thus

$$W(x) = \sum_{r \in R} U_r(x_r) - \sum_{j \in J} C_j\left(\sum_{s: j \in s} x_s\right)$$

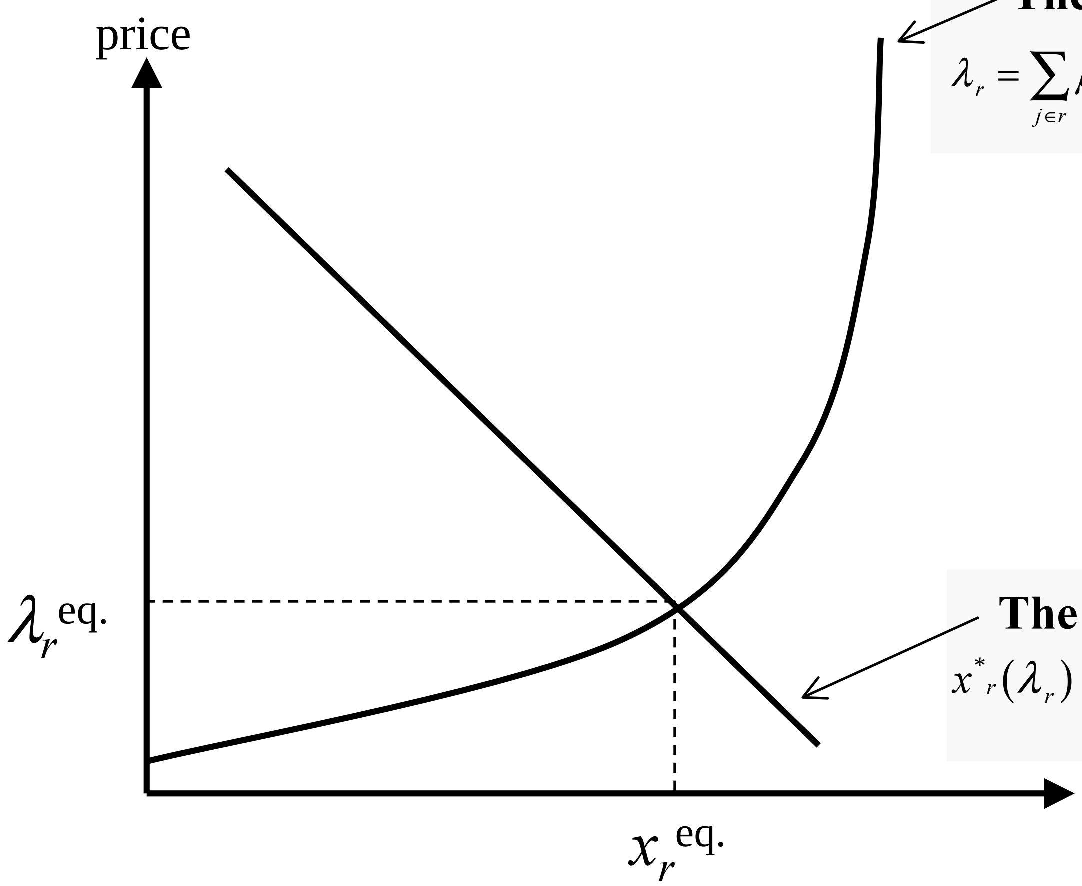
is a Lyapunov function for the system, the unique value maximising $W(x)$ is a stable point of the system

Reformulation (5)
Possible advantages

- We do not have to deal with utility functions per see, (as long as our business is to sell dynamically priced band-width, what is relevant for us is the demand function directed towards this good).

- Close to the classical formulation in economics (and thus there are meters of literature on different aspects (dynamics, optimality, imperfect competition etc etc.)).

The market illustrated



The supply function

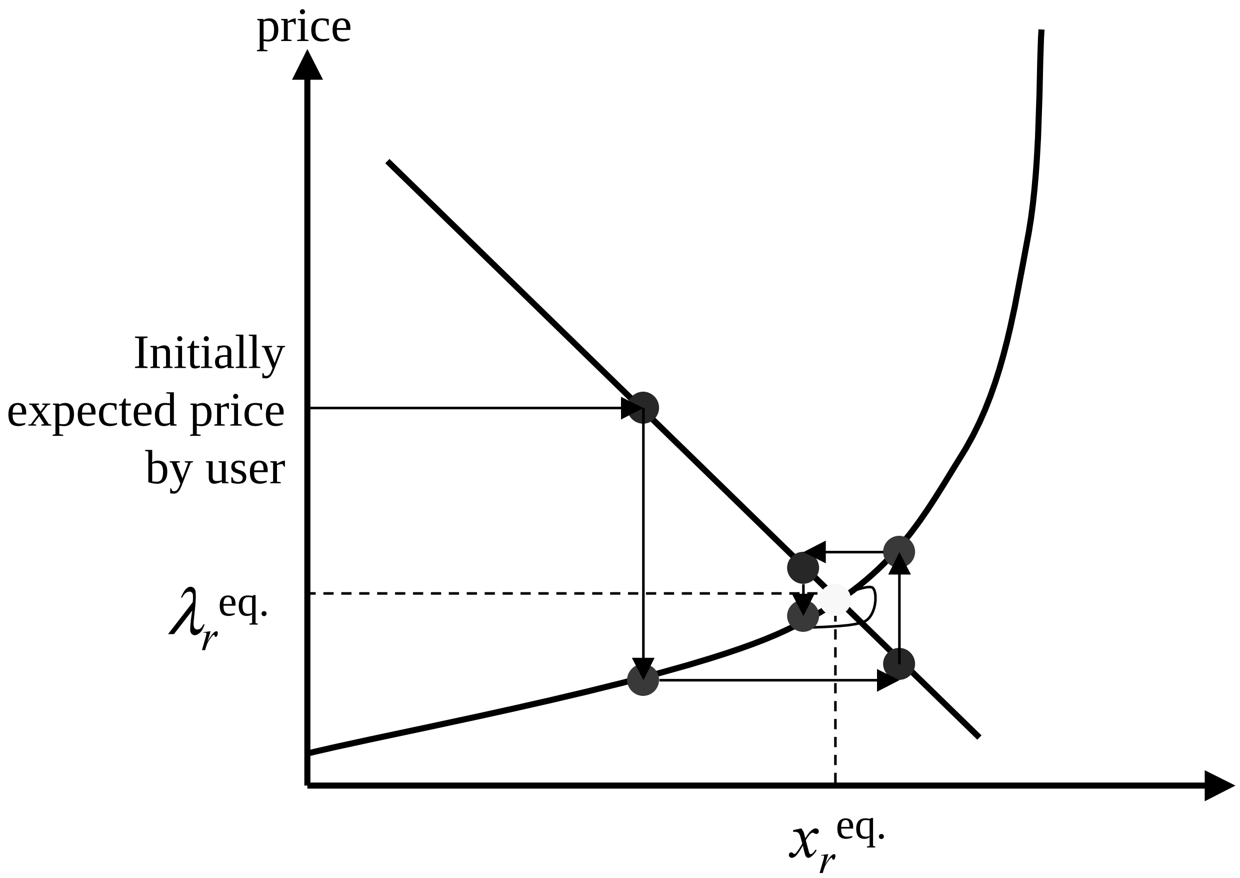
$$\lambda_r = \sum_{j \in r} \mu_j(t) = \sum_{j \in r} \left[p_j \left(\sum_{s: j \in s} x_s(t) \right) \right]$$

The demand function

$$x^*_r(\lambda_r) = \arg\max_{x_r} \left(U_r(x_r) - \lambda_r x_r \right)$$

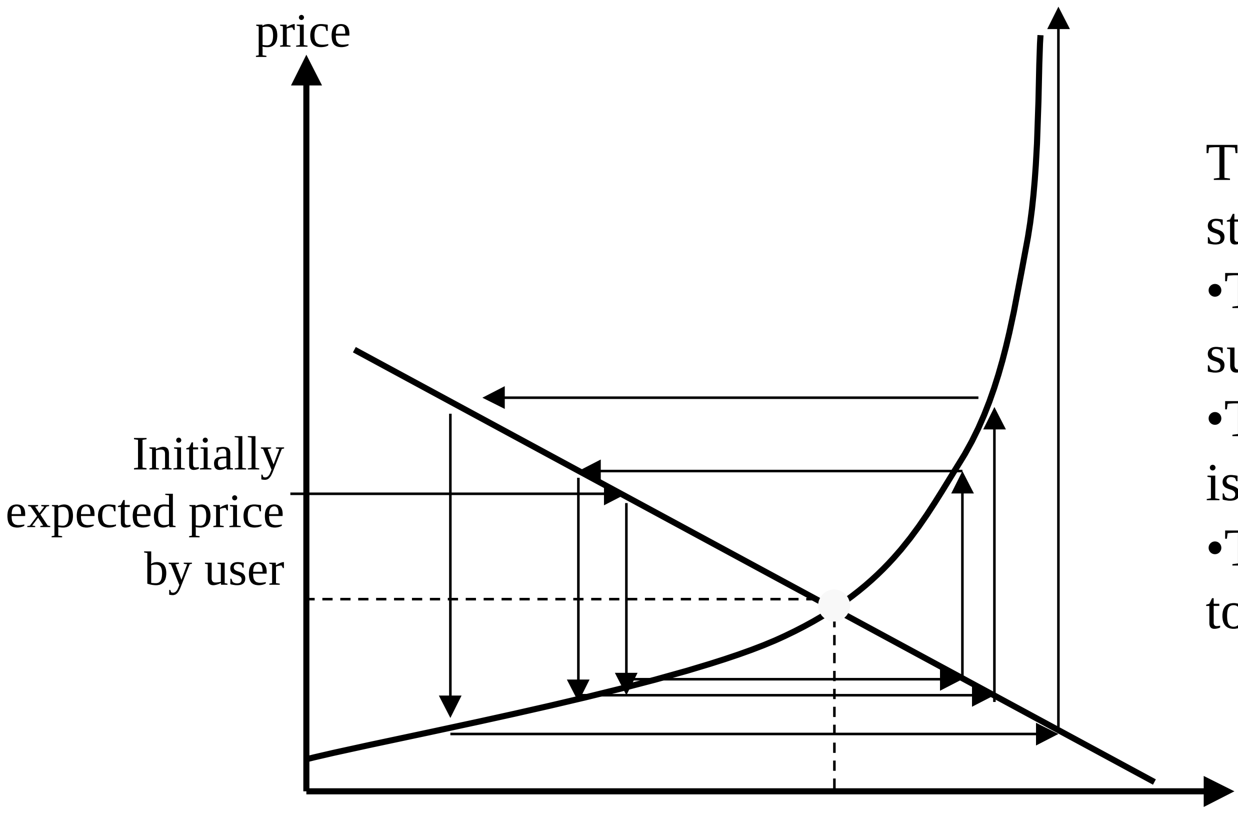
Stability (1)

Consider a simplified discrete model where the user have naive price expectations



Stability (2) ,"hog cycles"

The same example, changed slope of the demand function



The discrete system is stable if:

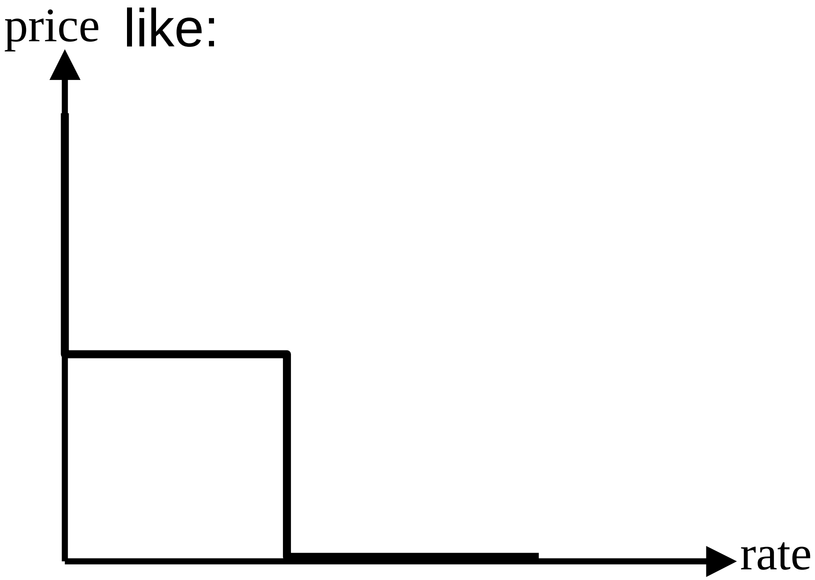
- The demand function is sufficiently steep
- The supply function not is to steep
- The parameter κ not is to large

As proved by Kelly et. al.:

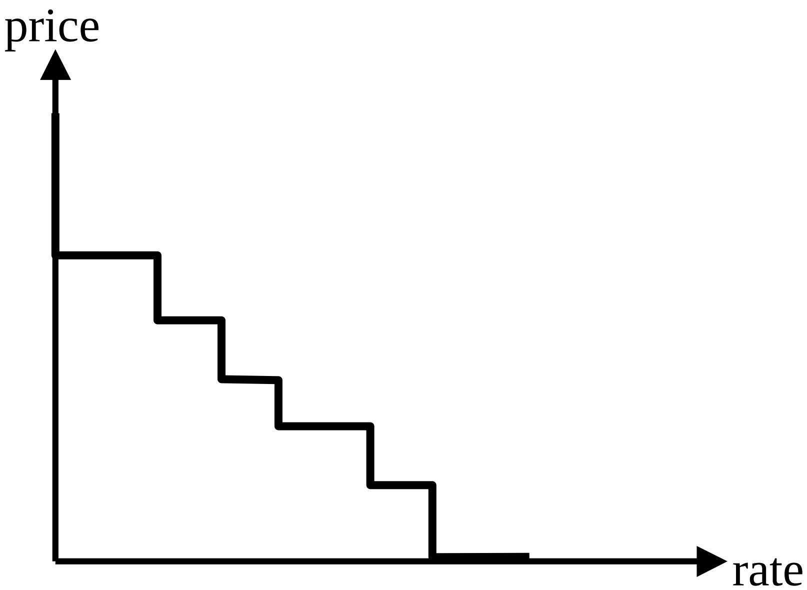
The continuous model is **always stable** when the demand function slopes downward, the supply function slope upwards and κ >0.

Inelastic traffic (1)

Consider a user with a fixed-rate application. This user will use the network if the price is below a threshold, Thus his demand function is like:



Five users with fixed-rate applications sorted by descending price threshold yields demand function:

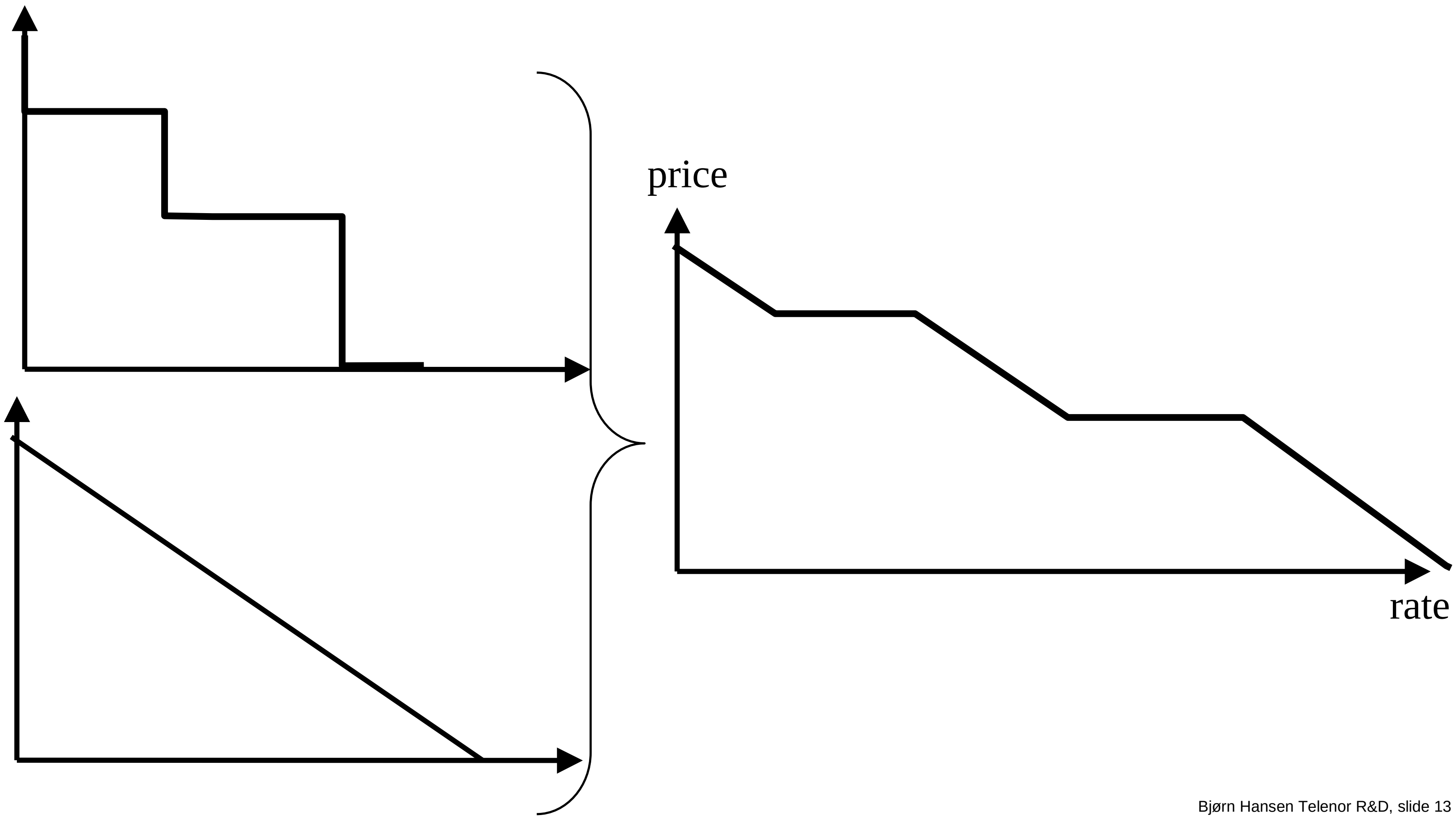


=> (almost) Nice downward sloping demand functions

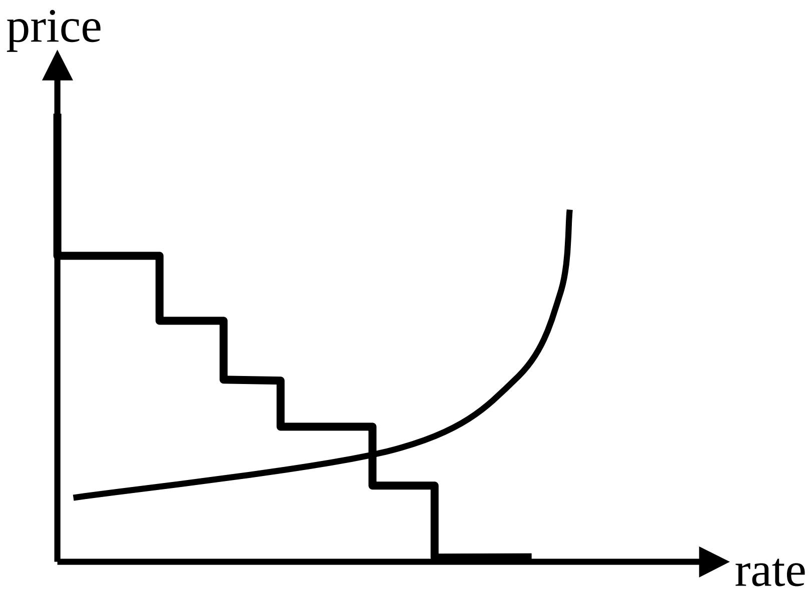
NB! these users can typically not use the rate control algorithm presented above

In-elastic traffic (2) Aggregation effects

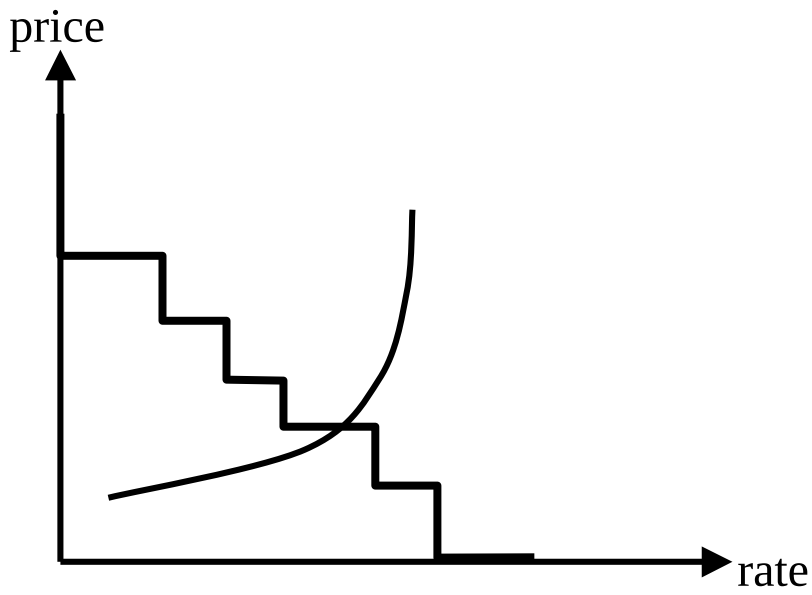
Example two in-elastic customers and one elastic



In-elastic traffic (3) Equilibrium



Equilibrium exists



Non existence of equilibrium

When the marginal inelastic users is small compared to the total load in each resource, there will generally be a price vector that results in demand being close to supply

The monopoly solution (1)

Problem: What is the optimal pricing rule for route r taking the cost and demand function into consideration

$$\Pi = \max_{\lambda_1,\ldots\lambda_r,\ldots} \left[\lambda_r x_r - \sum_{j \in r} C_j \left(\sum_{s: j \in s} x_s \right) \right] \quad \text{and} \quad x_r = x_r(\lambda_r)$$

$$\Downarrow$$

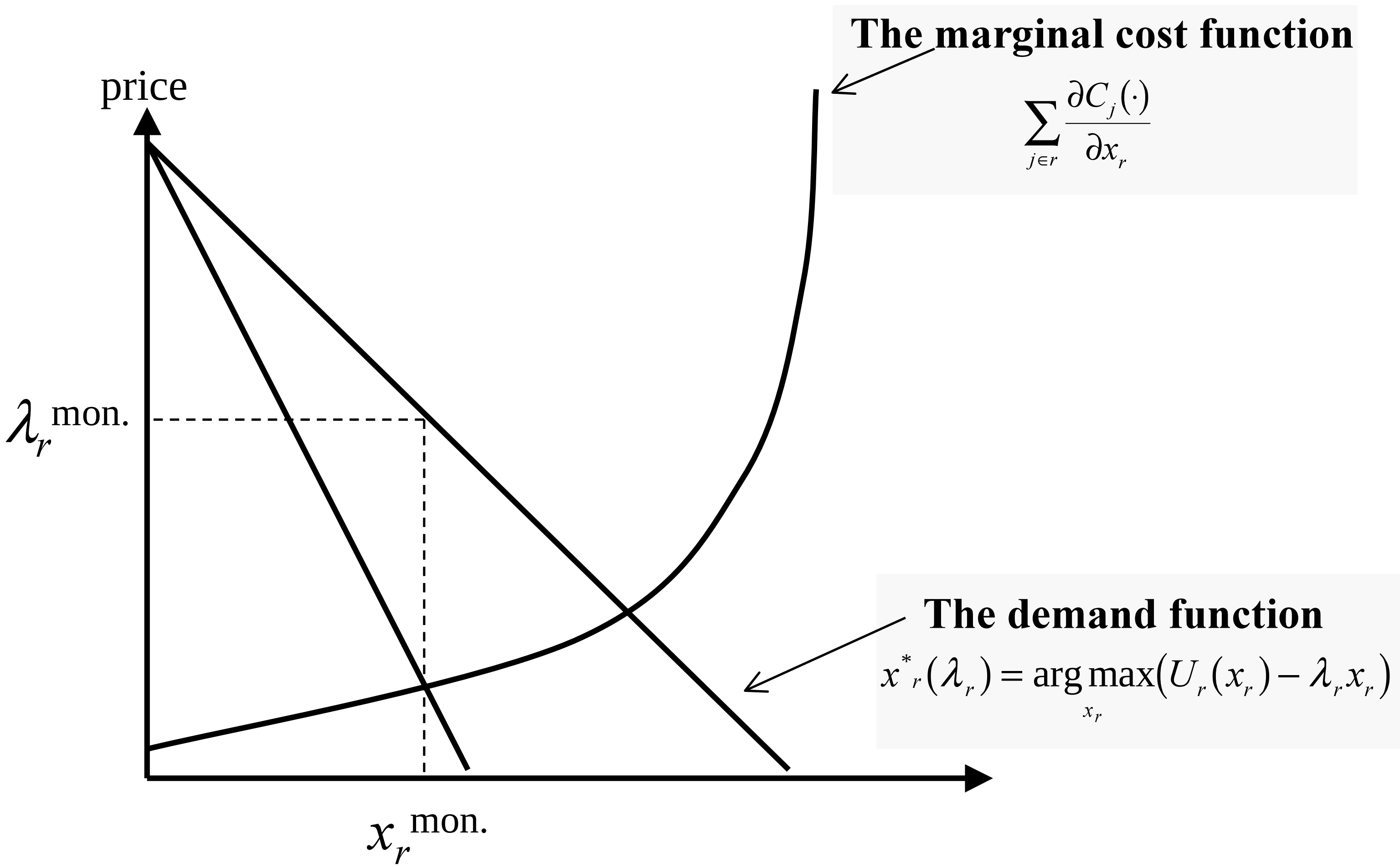
$$\Pi = \max_{\lambda_1,\ldots\lambda_r,\ldots} \left[\sum_{r \in R} \lambda_r x_r(\lambda_r) - \sum_{j \in J} C_j \left(\sum_{s: j \in s} x_s(\lambda_s) \right) \right]$$

First order condition:

$$x_r(\lambda_r) + \lambda_r \tfrac{\partial x_r}{\partial \lambda_r} - \sum_{j \in r} \frac{\partial C_j(\cdot)}{\partial x_r} \tfrac{\partial x_r}{\partial \lambda_r} = 0 \Leftrightarrow \sum_{j \in r} \frac{\partial C_j(\cdot)}{\partial x_r} = \lambda_r + x_r(\lambda_r) \frac{1}{\frac{\partial x_r}{\partial \lambda_r}}$$

i.e. marginal cost equal to marginal revenue

The monopoly solution (2)



The monopoly solution (3)

Consider a demand function jumping around with constant slope $-a$

Optimal pricing:

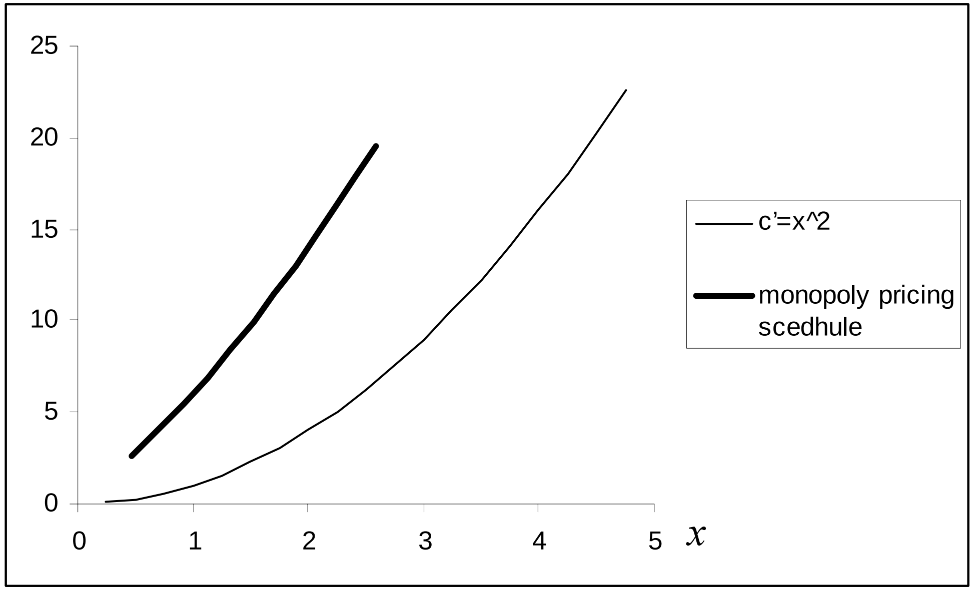
$$\lambda_r = \sum_{j \in r} \frac{\partial C_j(\cdot)}{\partial x_r} + x_r(\lambda_r)\frac{1}{a}$$

Numerical example:

$$\frac{d}{dx}C(x) = x^2$$

$$Demand: \quad p = A - ap$$

Where a = 5, A is shifting:



Perfect competition

Perfect competition (def) no supplier can influence the equilibrium price, thus the demand function (towards an arbitrary supplier) is horizontal =>

$$1 \Big/ \frac{\partial x_r}{\partial \lambda_r} = 0$$

Inserting this in the (monopoly) first order condition yields:

$$\lambda_r = \sum_{j \in r} \frac{\partial C_j(\cdot)}{\partial x_r} + x_r(\lambda_r) \frac{1}{\frac{\partial x_r}{\partial \lambda_r}}$$

$$\lambda_r = \sum_{j \in r} \frac{\partial C_j(\cdot)}{\partial x_r}$$

Identical to the solution studied by Kelly
NB: if there are externalities this is not first best

Externalities (1)

The actions of one agent directly affect the environment of another agent

- Network externalities: Value increase in the size (usage) of the network
- Congestion externalities: Value decrease in usage of the network due to delays and dropped packets

Externalities (2)
Congestion externalities

Assume one single resource, many users

- Let there be a constant negative congestion effect in the utility function (due to delay, packet loss, jitter, etc)

$$U_i(x_i, X) - x_i\lambda \quad \text{where} \quad X = \sum_j x_j$$

$$\frac{\partial U_i}{\partial x_i} > 0, \frac{\partial^2 U_i}{\partial x^2{}_i} < 0 \text{ etc,} \quad \frac{\partial U_i}{\partial X} < 0$$

- Market user behavior is almost the same as earlier:

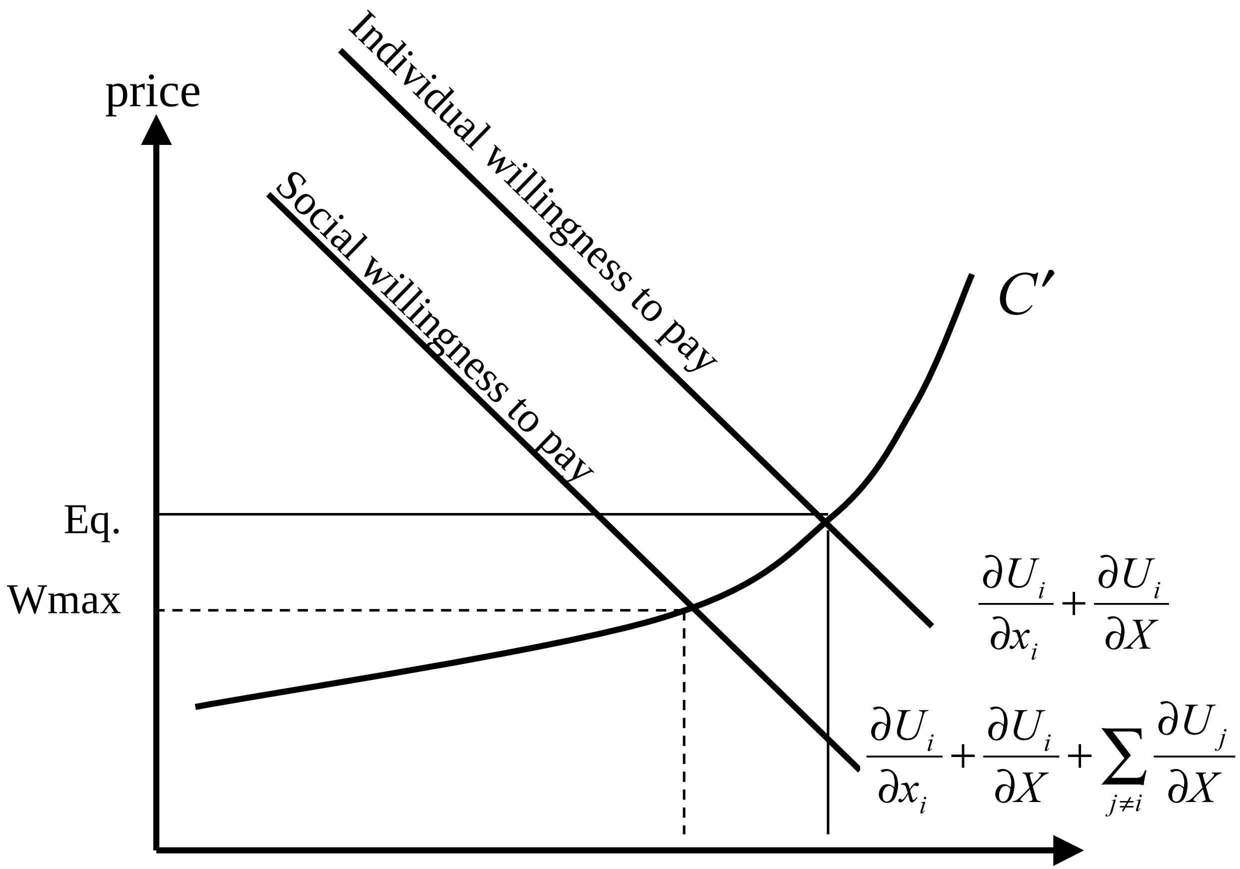
$$x^*{}_i(\lambda) = \arg\max_{x_i}\left(U_i(x_i, X) - \lambda x_i\right), \quad \text{FoC:} \qquad \frac{\partial U_i}{\partial x_i} + \frac{\partial U_i}{\partial x_i} = \lambda$$

- Maximising social welfare yields:

$$W = \sum_i U_i\left(x_i, \sum_j x_j\right) - C\left(\sum_j x_j\right)$$

$$\text{Foc:} \quad \frac{\partial U_i}{\partial x_i} + \frac{\partial U_i}{\partial X} + \sum_{j \neq i} \frac{\partial U_j}{\partial X} = C'$$

Externalities (3)
Congestion externalities illustrated



Some ideas for further studies

- Imperfect competition (e.g. duopoly) how do the suppliers interact
  - Offering substitutes (serving the same set of routes)
  - Offering complements (interconnection)
- Non linear tariffs, implementing product diferentiation by using a marginal price per packet differentiated over segments
- Network dimensioning two stage procedure, stage 1 investments in capacity, at stage 2 market solution